ON THE ASYMPTOTIC PROPERTIES OF THE GENERALIZED EXPONENTIAL FUNCTION OVER ISOLATED TIME SCALES

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A time scale is a nonempty, closed subset of $\mathbb{R}$ denoted by $\mathbb{T}$ for which a calculus can be developed. We define the solution to $y^{\Delta} = p(t)y$ to be the generalized exponential function. In this work, we consider the generalized exponential function $e_z(t, t_0)$ on isolated time scales where the exponent z is a complex number. We show several asymptotic properties of the generalized exponential function. In particular, we prove several theorems that define the regions of convergence of the exponential $e_z(t, t_0)$ to 0 as well as regions of divergence in the complex plane.

Contents

Chapter 1

Time Scale Calculus

1.1 Preliminary Time Scale Definitions

We will give a basic introduction to the time scale calculus touching on the topics that will be needed later. For a more thorough explanation, see [1].

Definition 1.1.1 (Time Scale)**.** *A time scale is a nonempty, closed subset of* $\mathbb{R}$ *denoted by* $\mathbb{T}$*.*

Definition 1.1.2 (Forward and Backward Jump Operators)**.** *Let* $\mathbb{T}$ *be a time scale, then for* $t \in \mathbb{T}$*, we define the forward jump operator* $\sigma(t) : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and define the backward jump operator $\rho(t) := \mathbb{T} \to \mathbb{T}$ *by*

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where it is understood that $\inf \emptyset = \sup \mathbb{T}$ *and* $\sup \emptyset = \inf \mathbb{T}$*.*

Definition 1.1.3 (Isolated Time Scale)**.** *A time scale* $\mathbb{T}$ *is said to be isolated if* $t_0 = \inf \mathbb{T} > -\infty$*, thus* $\sigma(t_0) > t_0$*, if* $t_0 = \sup \mathbb{T} < \infty$*, thus* $\rho(t_1) < t_1$*, and if* $\inf \mathbb{T} < t < \sup \mathbb{T}$ *it follows that* $\rho(t) < t < \sigma(t)$*.*

Definition 1.1.4 (Graininess Function)**.** *The graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\mu(t) := \sigma(t) - t.$$

1.2 The Generalized Exponential Function

Definition 1.2.1 (Regressive)**.** *A function* $p(t)$ *is said to be regressive if* $1+\mu(t)p(t) \neq 0$ *for all* $t \in \mathbb{T}$ *and is denoted by* $p \in \mathcal{R}$*.*

Definition 1.2.2 (Cylinder Transformation)**.** *The cylinder transformation* $\xi_h : \mathbb{C} \backslash \left\{-\frac{1}{h}\right\} \to \mathbb{C}$ *is defined as*

$$\xi_h(z) = \begin{cases} \frac{1}{h} \operatorname{Log}\left(1 + zh\right), & h > 0, \\ z, & h = 0, \end{cases}$$

where Log *is the principal logarithm function.*

Definition 1.2.3 (Generalized Exponential Function)**.** *For* $p \in \mathcal{R}$*, the generalized exponential function* $e_p : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *is defined by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

for $s, t \in \mathbb{T}$*.*

Theorem 1.2.1. *Given* $p \in \mathcal{R}$*, the solution to the first order dynamic equation*

$$y^{\Delta} = p(t)y$$

, with $y(t_0) = 1$*, is given by* $e_p(t, t_0)$*.*

Chapter 2

Asymptotic Properties of the Generalized Exponential Function

2.1 Introduction

We are concerned about the asymptotic properties of the generalized exponential function $e_z(t, t_0)$ where z is complex and regressive on isolated time scales that are unbounded above. That is $\mathbb{T} = \{t_0, t_1, \cdots\}$ where t_0 is fixed, $t_0 < t_1 < \cdots$, and $\sup \mathbb{T} = \infty$. In particular, we are concerned with the asymptotic behavior of $e_z(t_n, t_0)$ for $z \in \mathbb{C} \cap \mathcal{R}$.

It has been well established that if the time scale is isolated with a constant value for the graininess, say μ', then the region of convergence of the exponential $e_z(t, t_0)$ to 0 for $z \in \mathbb{C} \cap \mathcal{R}$ is given by

$$|1 + \mu' z| < 1.$$

Example 2.1.1. Consider the time scale generated by $\mu(t_n) = 1$ for all $n \in \mathbb{N}$. The region of convergence of the exponential $e_z(t, t_0)$ to 0 in $\mathbb{C} \cap \mathcal{R}$ is given by $|1 + z| < 1$.

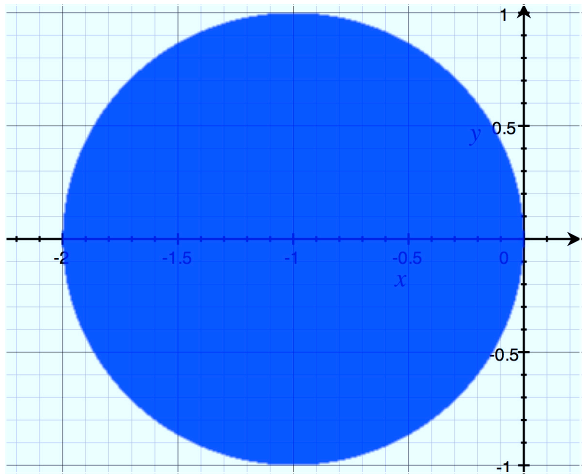

Figure 2.1: Region of convergence of the exponential $e_z(t, t_0)$ to 0 of the generalized exponential function in the complex plane for the time scale with a constant graininess of 1.

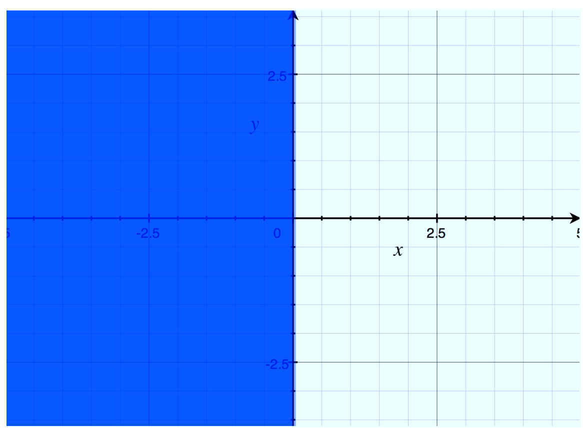

Figure 2.2: Region of convergence of the exponential $e_z(t, t_0)$ to 0 of the generalized exponential function when $\mathbb{T} = \mathbb{R}$.

This gives the region as a circle in the complex plane with centered at $(-1, 0)$ with a radius of 1, as seen in Figure 2.1.

On the other hand, if the time scale is the set of real numbers, i.e. the typical continuous case, we have that the region of convergence of the exponential $e_z(t, t_0)$ to 0 for $z \in \mathbb{C} \cap \mathcal{R}$ is given by $\text{Re}(z) < 0$.

This gives a region of the left half of the complex plane, seen in Figure 2.2.

Pötzsche et al. give the following theorem in [4] on the asymptotic behavior of the exponential for time scales that are unbounded above in the complex plane.

Theorem 2.1.1 ([4], Proposition 3.1)**.** *Assume* $\mathbb{T}$ *is unbounded above and fix* $t_0 \in \mathbb{T}$*. If for* $z \in \mathbb{C} \cap \mathcal{R}$ *such that*

$$\limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu(t)} \frac{\log|1 + zs|}{s} \Delta s < 0,$$

then $\lim_{n\to\infty} e_z(t_n, t_0) = 0$

This theorem is very difficult to work with due to the lim sup, as well as that it needs to take into account the entire structure of the time scale to calculate the Δ-integral as well as particular points in the time scale $\mathbb{T}$. The goal of this thesis is given an isolated, unbounded from above time scale $\mathbb{T}$, we want to find the region in the complex plane independent of $t \in \mathbb{T}$ such that $\lim_{n\to\infty} e_z(t_n, t_0) = 0$.

2.2 Preliminary Lemmas

This first lemma gives criteria to show either convergence to 0 or divergence of the generalized exponential function.

Lemma 2.2.1. *Assume* $\mathbb{T} = \{t_0, t_1, \cdots\}$ *where* $t_0 < t_1 < \cdots$ *and* $\sup \mathbb{T} = \infty$*, and let* $z \in \mathbb{C} \cap \mathcal{R}$ *be given.*

1: If $\sum_{k=0}^{\infty} \ln|1 + \mu(t_k)z| = -\infty$*, then* $\lim_{n\to\infty} e_z(t_n, t_0) = 0$*.*

2: If $\sum_{k=0}^{\infty} \ln|1 + \mu(t_k)z| = \infty$*, then* $\lim_{n\to\infty} e_z(t_n, t_0)$ *diverges.*

Proof. The generalized exponential function is defined by using the cylinder transform as

$$e_p(t,s) := \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ with } \xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0, \end{cases}$$

where Log is the principal logarithm.

Let $z \in \mathbb{C} \cap \mathcal{R}$. Then, since this is an isolated time scale, we have $\mu(t_n) \neq 0$, so the generalized exponential function is

$$e_z(t_n, t_0) = \exp\left(\sum_{k=0}^{n-1} \mu(t_k) \frac{\ln(1+\mu(t_k)z)}{\mu(t_k)}\right) = \exp\left(\sum_{k=0}^{n-1} \ln(1+\mu(t_k)z)\right).$$

We are concerned with the generalized exponential's behavior as $n \to \infty$, so we consider the modulus of the exponential as

$$\lim_{n\to\infty} |e_z(t_n, t_0)| = \exp\left(\sum_{k=0}^{\infty} \ln|1+\mu(t_k)z|\right).$$

Define $x_n := \sum_{k=0}^{n} \ln|1+\mu(t_k)z|$ and assume $\lim_{n\to\infty} x_n = -\infty$, i.e. $\sum_{k=0}^{\infty} \ln|1+\mu(t_k)z| = -\infty$. Then $\lim_{n\to\infty} e^{x_n} = 0$, so $\exp\left(\sum_{k=0}^{\infty} \ln|1+\mu(t_k)z|\right) = 0$, implying $e_z(t_n, t_0)$ converges to 0 as $n \to \infty$.

Similarly, to show the generalized exponential function diverges as $n \to \infty$, it is sufficient to show that $\sum_{k=0}^{\infty} \ln|1+\mu(t_k)z| = \infty$. □

With this lemma in hand, we now give two lemmas that show convergence of the exponential $e_z(t, t_0)$ to 0 or divergence based on the lim sup or lim inf of $|1+\mu(t_k)z|$ respectively. This allows us to deal with time scales where the graininess never repeats a value, but rather the values are formed by letting $\{\mu(t_n)\}t_{n=0}^{\infty}$ be a sequence with a nonzero limit as $n \to \infty$.

Lemma 2.2.2. *Assume* $\mathbb{T} = \{t_0, t_1, \cdots\}$ *where* $t_0 < t_1 < \cdots$ *and* $\sup \mathbb{T} = \infty$*, and let* $z \in \mathbb{C} \cap \mathcal{R}$ *be given. If* $0 < \limsup_{k\to\infty} |1 + \mu(t_k)z| < 1$*, then* $\lim_{n\to\infty} e_z(t_n, t_0) = 0$*.*

Proof. Let $z \in \mathbb{C} \cap \mathcal{R}$ be given such that $0 < \limsup_{k\to\infty} |1 + \mu(t_k)z| < 1$. Then $\limsup_{k\to\infty} \ln |1 + \mu(t_k)z| < 0$. Define $L := \limsup_{k\to\infty} \ln |1 + \mu(t_k)z|$. By definition, this implies that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\ln |1 + \mu(t_n)z| < \epsilon + L$. Note that for sufficiently small ϵ, $\sum_{k=0}^{\infty} (\epsilon + L) = -\infty$. Using the comparison test and noting $\ln |1 + \mu(t_n)z| < \epsilon + L < 0$ for $n \geq N$, we have $\sum_{k=0}^{\infty} \ln |1 + \mu(t_k)z| = -\infty$. By Lemma 2.2.1 we then have $\lim_{n\to\infty} e_z(t_n, t_0) = 0$.

□

Lemma 2.2.3. *Assume* $\mathbb{T} = \{t_0, t_1, \cdots\}$ *where* $t_0 < t_1 < \cdots$ *and* $\sup \mathbb{T} = \infty$*, and let* $z \in \mathbb{C} \cap \mathcal{R}$ *be given. If* $\liminf_{k\to\infty} |1 + \mu(t_k)z| > 1$*, then* $\lim_{n\to\infty} e_z(t_n, t_0)$ *diverges.*

Proof. Let $z \in \mathbb{C} \cap \mathcal{R}$ be given such that $\liminf_{k\to\infty} |1 + \mu(t_k)z| > 1$. This implies $\liminf_{k\to\infty} \ln |1 + \mu(t_k)z| > 0$. Define $L := \liminf_{k\to\infty} \ln |1 + \mu(t_k)z|$. By definition, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\ln |1 + \mu(t_n)z| > \epsilon + L > 0$. Note that $\sum_{k=0}^{\infty} \epsilon + L = \infty$. Thus by the comparison test, $\sum_{k=0}^{\infty} \ln |1 + \mu(t_k)z| = \infty$, so by Lemma 2.2.1, $e_z(t_n, t_0)$ diverges as $n \to \infty$.

□

Example 2.2.1. Consider $\mu(t_n) = 1 + \frac{1}{n^2}$. We have then $\limsup_{n\to\infty} \mu(t_n) = \liminf_{n\to\infty} \mu(t_n) = 1$. By Lemma 2.2.2, the generalized exponential function converges to 0 for $z \in \mathbb{C} \cap \mathcal{R}$ such that $|1 + 1 \cdot z| < 1$, and by Lemma 2.2.3 the exponential diverges when $|1 + 1 \cdot z| > 1$. This gives a circle of convergence to 0 of the exponential in the left half plane centered at $(-1, 0)$ with radius 1 as seen in Figure 2.3 and divergence for any point outside that circle.

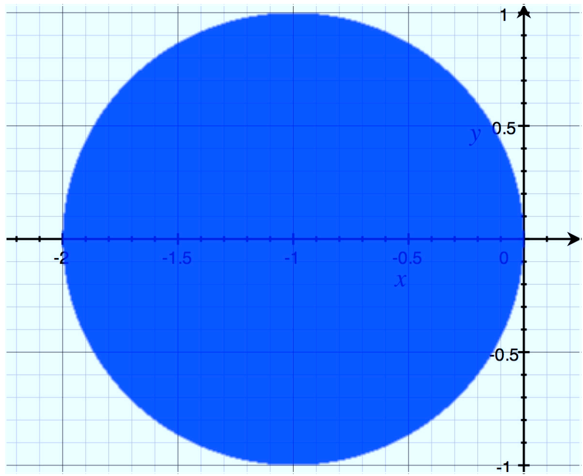

Figure 2.3: Region of convergence of the exponential $e_z(t, t_0)$ to 0 of the generalized exponential function in the complex plane for the time scale with the limit of the graininess being 1.

2.3 Time Scales with Repeated Value Grainінessess

Example 2.3.1. Consider the time scale defined by $t_0 = 0$ and

$$\mu(t_k) = \begin{cases} 1 & \text{if k is even} \\ 5 & \text{if k is odd.} \end{cases}$$

That is $t_0 = 0$, $t_1 = 1$, $t_2 = 6$, $t_3 = 7$, $t_4 = 12$, $\cdots$. From [2] and [3], for this time scale, the generalized exponential function converges to 0 if $|1 + 1 \cdot z|\,|1 + 5 \cdot z| < 1$ as seen in Figure 2.4.

This alternating graininess concept can be extended. The first extension is to

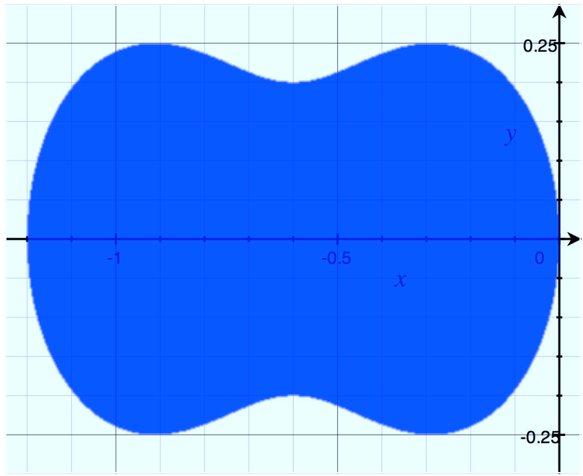

Figure 2.4: Region of convergence of the exponential $e_z(t, t_0)$ to 0 of the generalized exponential function in the complex plane for the time scale with graininesses alternating between 1 and 5.

have more than two graininesses, i.e. for some $N \in \mathbb{N}$,

$$\mu(t_k) = \begin{cases} r_0 & \text{if } k \equiv 0 \mod N \\ r_1 & \text{if } k \equiv 1 \mod N \\ \vdots & \\ r_{N-1} & \text{if } k \equiv N-1 \mod N, \end{cases}$$

where $r_i > 0$ for all $i = 0, \ldots, N-1$.

This following proposition generalizes the alternating graininess case to N repeated graininesses.

Proposition 2.3.1. *Assume* $\mathbb{T} = \{t_0, t_1, \cdots\}$ *where* $t_0 < t_1 < \cdots$ *and* $\sup \mathbb{T} = \infty$, *and that there exists an* $N \in \mathbb{N}$ *such that for all* $0 \leq i \leq N-1$ *there exists a real number* $r_i > 0$ *such that* $\mu(t_{n_k^i}) = r_i$ *where* $\{n_k^i\}_{k=0}^{\infty}$ *is the subsequence of* $\{t_n\}_{n=0}^{\infty}$ *such that* $n_k^i \equiv i \mod N$ *for all* $k \in \mathbb{N}$. *Let* $z \in \mathbb{C} \cap \mathcal{R}$ *be given.*

1: If $0 < \prod_{j=0}^{N-1} |1 + r_j z| < 1$*, then* $\lim_{n\to\infty} e_z(t_n, t_0) = 0$.

2: If $\prod_{j=0}^{N-1} |1 + r_j z| > 1$*, then* $\lim_{n\to\infty} e_z(t_n, t_0)$ *diverges.*

Proof. Proof of (1): Let $N \in \mathbb{N}$ and assume there exists $r_i > 0$ for all $0 \leq i \leq N-1$ where $\mu(t_{n_k^i}) = r_i$. We consider the subsequence $n_k^0 = kN$ for each $k \in \mathbb{N}$. We then have

$$|e_z(t_{kN}, t_0)| = \left(|1 + r_0 z|\, |1 + r_1 z| \cdots |1 + r_{N-1} z|\right)^k .$$

Then $\lim_{k\to\infty} |e_z(t_{kN}, t_0)| = 0$ if

$$|1 + r_0 z|\, |1 + r_1 z| \cdots |1 + r_{N-1} z| < 1.$$

Now we consider the subsequence $n_k^1 = kN + 1$ for each $k \in \mathbb{N}$. We then have

$$|e_z(t_{kN+1}, t_0)| = |1 + r_0 z| \left(|1 + r_0 z|\, |1 + r_1 z| \cdots |1 + r_{N-1} z|\right)^k .$$

Again, $\lim_{k\to\infty} |e_z(t_{kN+1}, t_0)| = 0$ if

$$|1 + r_0 z|\, |1 + r_1 z| \cdots |1 + r_{N-1} z| < 1.$$

We continue in this fashion until we reach the subsequence $n_k^{N-1} = kN + N - 1$, where we have

$$\left|e_z(t_{kN+(N-1)}, t_0)\right| = |1 + r_0 z|\, |1 + r_1 z| \cdots |1 + r_{N-2}| \left(|1 + r_0 z|\, |1 + r_1 z| \cdots |1 + r_{N-1}|\right)^k ,$$

which also converges to 0 if

$$|1 + r_0 z|\, |1 + r_1 z| \cdots |1 + r_{N-1} z| < 1.$$

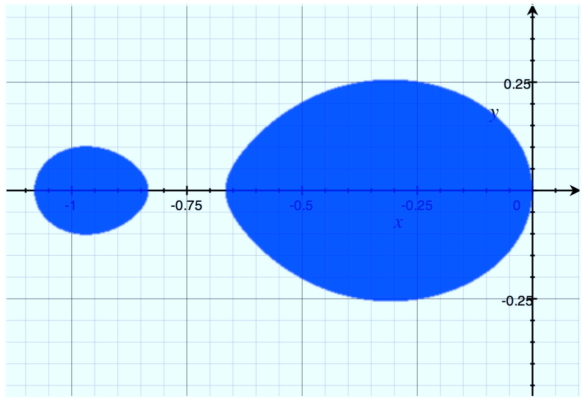

Figure 2.5: Region of convergence of the exponential $e_z(t, t_0)$ to 0 of the generalized exponential function in the complex plane for the time scale with the value of the graininess cycling 1, 3, and 5.

Since these subsequences account for every term in the original sequence t_n, we have if

$$|1 + r_0 z|\,|1 + r_1 z| \cdots |1 + r_{N-1} z| = \prod_{j=0}^{N-1} |1 + r_j z| < 1,$$

then

$$\lim_{n\to\infty} e_z(t_n, t_0) = 0.$$

The proof of (2) follows similarly by switching the direction of the inequality signs. $\square$

Example 2.3.2. Consider the time scale defined by the graininess which is defined by

$$\mu(t_k) = \begin{cases} 1 & \text{if } k \equiv 0 \mod 3 \\ 3 & \text{if } k \equiv 1 \mod 3 \\ 6 & \text{if } k \equiv 2 \mod 3. \end{cases}$$

The region of convergence of the exponential $e_z(t, t_0)$ to 0 in the complex plane is given by $|1 + z|\,|1 + 3z|\,|1 + 6z| < 1$ as seen in Figure 2.5.

A second extension is that instead of having the graininess values take on N constants, we have N sequences with nonzero limits.

Example 2.3.3. Consider the time scale defined by a graininess sequence such that

$$\mu(t_k) = \begin{cases} \frac{1}{(k+1)^2} + 1 & \text{if k is even} \\ \\ \frac{1}{k} + 5 & \text{if k is odd,} \end{cases}$$

then the following theorem will show that this case results in the exact same region of convergence of the exponential $e_z(t, t_0)$ to 0 shown in Figure 2.4 where the graininess alternates in value between constant values 1 and 5.

Theorem 2.3.1. *Assume* $\mathbb{T} = \{t_0, t_1, \cdots\}$ *where* $t_0 < t_1 < \cdots$ *and* $\sup \mathbb{T} = \infty$, *and that there exists an* $N \in \mathbb{N}$ *such that for all* $0 \leq i \leq N-1$ *there exists a real number* $r_i > 0$ *such that* $\lim_{k\to\infty} \mu(t_{n_k^i}) = r_i$ *where* $\{t_{n_k^i}\}_{k=0}^{\infty}$ *is the subsequence of* $\{t_n\}_{n=0}^{\infty}$ *such that* $n_k^i \equiv i \mod N$ *for all* $k \in \mathbb{N}$. *Let* $z \in \mathbb{C} \cap \mathcal{R}$ *be given.*

1: If $0 < \prod_{j=0}^{N-1} |1 + r_j z| < 1$, *then* $\lim_{n\to\infty} e_z(t_n, t_0) = 0$.

2: If $\prod_{j=0}^{N-1} |1 + r_j z| > 1$, *then* $\lim_{n\to\infty} e_z(t_n, t_0)$ *diverges.*

Proof. Proof of (1): Let $\epsilon > 0$ and $N \in \mathbb{N}$ be given. Assume there exists $r_i > 0$ such that for all $0 \leq i \leq N-1$, $\lim_{k\to\infty} \mu(t_{n_k^i}) = r_i$. Finally assume $\prod_{j=0}^{N-1} |1 + r_j z| < 1$, which implies that there exists $0 < \delta < 1$ such that $\prod_{j=0}^{N-1}(\delta + |1 + r_j z|) < 1$.

Because $\lim_{k\to\infty} \mu(t_{n_k^i}) = r_i$, we have $\lim_{k\to\infty} \left|1 + \mu(t_{n_k^i})z\right| = |1 + r_i z|$. This implies that there exists M_i's for each $0 \leq i \leq N-1$ such that for all $k > M_i$, $\left|1 + \mu(t_{n_k^i})z\right| < \delta + |1 + r_i z|$. Let

$$L_0 = \max\left\{ \left(\prod_{j=0}^{k} (\delta + |1 + r_j z|) \right)_{k=0}^{N-1} \right\}$$

Furthermore, since $\prod_{j=0}^{N-1}(\delta + |1 + r_j z|) < 1$, there exists some $L \in \mathbb{N}$ such that for all $k > L$,

$$L_0 \left(\prod_{j=0}^{N-1} (\delta + |1 + r_j z|) \right)^k < \epsilon.$$

Let $M \geq \max\{\{M_i\}_{i=0}^{N-1}, L\}$ such that $M \equiv 0 \mod N$.

Consider the subsequence $n_k^0 = M + kN$. We have that for $k \in \mathbb{N}$,

$$|e_z(t_{M+kN}, t_M)| < \left(\prod_{j=0}^{N-1} (\delta + |1 + r_j z|) \right)^{M+k} < \epsilon.$$

Thus $\lim_{k\to\infty} |e_z(t_{M+kN}, t_M)| = 0$ if $\prod_{j=0}^{N-1}(\delta + |1 + r_j z|) < 1$.

Consider the subsequence $n_k^1 = M + kN + 1$. We have that for $k \in \mathbb{N}$,

$$|e_z(t_{M+kN+1}, t_M)| < (\delta + |1 + r_0 z|) \left(\prod_{j=0}^{N-1} (\delta + |1 + r_j z|) \right)^{M+k} < \epsilon.$$

Thus $\lim_{k\to\infty} |e_z(t_{M+kN+1}, t_M)| = 0$ if $\prod_{j=0}^{N-1}(\delta + |1 + r_j z|) < 1$.

We continue on until we consider $n_k^{N-1} = M + kN + (N - 1)$. Then for all $k \in \mathbb{N}$,

$$\left|e_z(t_{M+kN+(N-1)}, t_M)\right| < \left(\prod_{j=0}^{N-2} (\delta + |1 + r_j z|) \right) \left(\prod_{j=0}^{N-1} (\delta + |1 + r_j z|) \right)^{M+k} < \epsilon.$$

Thus $\lim_{k\to\infty} \left|e_z(t_{M+kN+(N-1)}, t_M)\right| = 0$ if $\prod_{j=0}^{N-1}(\delta + |1 + r_j z|) < 1$.

Since the subsequences $\{n_k^i\}_{k=M}^{\infty}$ for $0 \leq i \leq N - 1$ include every point in the original sequence $\{\mu(t_n)\}_{n=M}^{\infty}$ we have that $\lim_{n\to\infty} e_z(t_n, t_M) = 0$ if $\prod_{j=0}^{N-1} |1 + r_j z| < \prod_{j=0}^{N-1}(\delta + |1 + r_j z|) < 1$. Since we are considering the asymptotic behavior of $e_z(t, t_0)$,

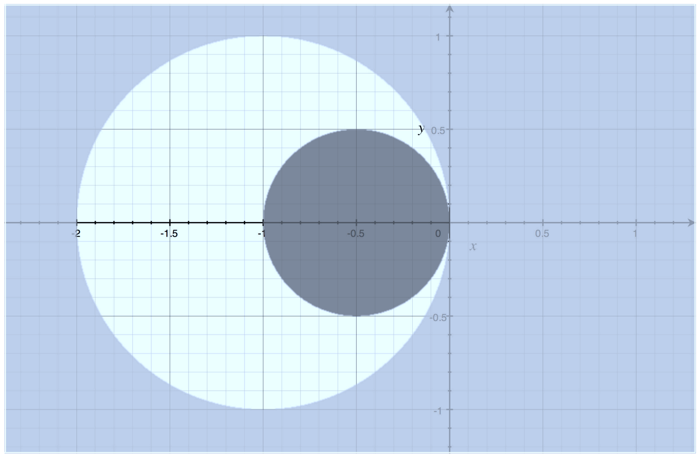

Figure 2.6: The dark gray region is a region of convergence of the exponential $e_z(t, t_0)$ to 0 of the generalized exponential function, the light gray region is a region of divergence, and the white region is indeterminate.

this implies if for $z \in \mathbb{C} \cap \mathcal{R}$ such that

$$\prod_{j=0}^{N-1} |1 + r_j z| < 1,$$

then

$$\lim_{n\to\infty} |e_z(t_n, t_0)| = 0.$$

The proof of (2) follows similarly by changing the direction of the inequalities. $\square$

2.4 Regular Patterns of Two Value Graininesses

Example 2.4.1. Consider the time scale $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$ where $t_0 = 0$ and $(\mu(t_n))_0^\infty = (1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, \cdots)$. Note this does not fall into the previous case with alternating graininesses, because there is no fixed length pattern that repeats. Thus we want to apply our preliminary lemmas, so we need to calculate $\limsup_{k\to\infty} |1 + \mu(t_k)z|$ and $\liminf_{k\to\infty} |1 + \mu(t_k)z|$. Let $z \in \mathbb{C} \cap \mathcal{R}$ such that

$\mathrm{Re}(z) < 0$ be given. Note

$$|1 + \mu(t_k)z| = \begin{cases} |1+z|, & k = \frac{1}{2}n(n+1) \text{ for } n \in \mathbb{N} \\ |1+2z|, & \text{otherwise.} \end{cases}$$

To find the lim inf and lim sup we must consider the question: is $|1+z| \geq |1+2z|$ or $|1+z| \leq |1+2z|$? Consider the linear fractional mapping

$$F(z) = \frac{1+z}{1+2z}.$$

This maps all points inside the circle of radius $\frac{1}{3}$ centered at $(-\frac{1}{3}, 0)$, i.e.

$$\left| z + \frac{1}{3} \right| < \frac{1}{3},$$

to points outside the unit circle, as well as mapping all points outside the unit circle to inside the circle of radius $\frac{1}{3}$ centered at $(-\frac{1}{3}, 0)$. We also have that for z such that, $\left|z + \frac{1}{3}\right| = \frac{1}{3}$, it is mapped onto the unit circle. We then have two cases.

Case 1: Assume $z \in \mathbb{C} \cap \mathcal{R}$, such that $\left|z + \frac{1}{3}\right| < \frac{1}{3}$. We have

$$|F(z)| = \left| \frac{1+z}{1+2z} \right| > 1,$$

thus

$$|1+z| > |1+2z|.$$

This gives

$$\limsup_{k \to \infty} |1 + \mu(t_k)z| = |1+z|.$$

Since $\left|z+\frac{1}{3}\right|<\frac{1}{3}$, we have $|1+z|<1$, thus for z in this case, the generalized exponential function converges to 0 by Lemma 2.2.1.

Case 2: Assume $z \in \mathbb{C} \cap \mathcal{R}$, such that $\left|z+\frac{1}{3}\right|>\frac{1}{3}$. We have

$$|F(z)| = \left|\frac{1+z}{1+2z}\right| < 1$$

which implies

$$|1+z| < |1+2z|.$$

This implies

$$\limsup_{k\to\infty} |1+\mu(t_k)z| = |1+2z|,$$

and

$$\liminf_{k\to\infty} |1+\mu(t_k)z| = |1+z|.$$

By applying Lemma 2.2.2 we know that the generalized exponential function converges to 0 for $z \in \mathbb{C} \cap \mathcal{R}$ such that $|1+2z|<1$, and we know that it diverges for $z \in \mathbb{C} \cap \mathcal{R}$ such that $|1+z|>1$ by Lemma 2.2.3.

The white space in Figure 2.4 has undetermined behavior. For $k \in \mathbb{N}$, we consider

$$\left|e_z\left(t_{\frac{k(k+1)}{2}}, t_0\right)\right| = |1+z|^k\,|1+2z|^{\frac{k(k-1)}{2}}.$$

Note

$$\begin{aligned}
|1+z|^k\,|1+2z|^{\frac{k(k-1)}{2}} &= |1+z|^k\,|1+2z|^{\frac{k^2}{2}-\frac{k}{2}}\\
&= |1+z|^k\,\frac{\sqrt{|1+2z|}^{k^2}}{\sqrt{|1+2z|}^{k}}\\
&= \left(\frac{|1+z|}{\sqrt{|1+2z|}}\right)^k \sqrt{|1+2z|}^{k^2}.
\end{aligned}$$

Since z is a complex number, we can define $a := \frac{|1+z|}{\sqrt{|1+2z|}}$ and $b := \sqrt{|1+2z|}$ where $a, b \in \mathbb{R}$. Let us consider $b > 1$, that is $\sqrt{|1+2z|} > 1$ which implies $|1+2z| > 1$. Then $\lim_{n\to\infty}(a^k b^{k^2})$ will give us the behavior of the generalized exponential function for this particular subsequence.

Claim: If $a \in \mathbb{R}$ and $b > 1$, then $\lim_{k\to\infty}(a^k b^{k^2}) = \infty$.

Assume to the contrary that there exists some $M \in \mathbb{R}$ such that

$$a^k b^{k^2} < M,$$

for all k. Then we have

$$\begin{aligned}
(b^k)^n &< Ma^{-k}\\
b^k &< \frac{M^{1/k}}{a}\\
\lim_{k\to\infty} b^k &\le \lim_{k\to\infty} \frac{M^{1/k}}{a}.
\end{aligned}$$

By assumption, $b > 1$, so the left hand side approaches infinity as $k \to \infty$, but the right hand side approaches $\frac{1}{a}$ as $k \to \infty$ as M and b are just some constants. This is a contradiction, as $\infty \not< \frac{1}{a}$. Thus $a^k b^{k^2}$ diverges to infinity.

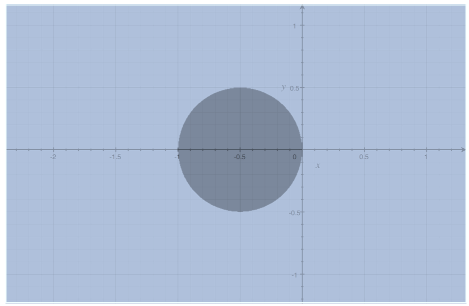

Figure 2.7: The dark gray region is the region of convergence to 0 of the generalized exponential function and the light gray region is the region of divergence.

This shows that the generalized exponential function diverges for this particular subsequence when $b = |1 + 2z| > 1$. Then, since one subsequence diverges to infinity, we have, in general, for $z \in \mathbb{C} \cap \mathcal{R}$ such that $|1 + 2z| > 1$,

$$\lim_{n\to\infty} |e_z(t_n, t_0)| = \infty.$$

This then shows that in Figure 2.4, the only region of convergence is the dark gray region, that is $|1 + 2z| < 1$.

Remark 2.4.1. *It is interesting in this example that while both 1 and 2 appear an infinite number of times as the values of the graininess, the number of 2s as the value in the graininess "overpowers" the number of 1s due to it appearing much more frequently.*

To generalize this result, we introduce the graininess counting function. We define $C_i(n)$ to be the count of the number of times the graininess of the time scale is i in the first n points.

Definition 2.4.1 (Graininess Counting Function)**.** *For some* $i > 0$*,*

$$C_i(n) := |\{\mu(t_k)|\mu(t_k) = i \text{ for } k = 0, \dots, n\}|,$$

where the vertical bars represent the cardinality of the set.

Note $C_i(n)$ increases to infinity with respect to n.

So in the previous example, we had $C_1(\frac{k(k+1)}{2}) = k$ and $C_2(\frac{k(k+1)}{2}) = \frac{k(k-1)}{2}$ for $k \in \mathbb{N}$.

Theorem 2.4.1. *Let $\mathbb{T}$ be an isolated time scale with two graininesses, a and b, occurring in the time scale infinitely many times, and let their counting functions be $C_a(n) = f(n)$ and $C_b(n) = g(n)$. Let $z \in \mathbb{C} \cap \mathcal{R}$. If $\lim_{k\to\infty} \frac{f(k)}{g(k)} = \infty$ and if $|1 + az| > 1$, then*

$$\lim_{n\to\infty} |e_z(t_n, t_0)| = \infty.$$

Proof. Let a, b, $f(n)$, $g(n)$, and z be given as in the theorem statement. Let $A = |1 + az| > 1$ and $B = |1 + bz|$. The generalized exponential function is then

$$|e_z(t_n, t_0)| = A^{f(n)} B^{g(n)}.$$

Since

$$\lim_{k\to\infty} \frac{f(k)}{g(k)} = \infty,$$

for all $M > 0$, there exists an $N \in \mathbb{N}$ such that $f(n) > Mg(n)$ for all $n \geq N$. We then have

$$\begin{aligned} f(n) &> Mg(n) \\ A^{f(n)} &> A^{Mg(n)} \\ A^{f(n)} B^{g(n)} &> A^{Mg(n)} B^{g(n)} = (A^M B)^{g(n)} \end{aligned}$$

We want to show $(A^M B) > 1$, as then we would have $\lim_{n\to\infty} (A^M B)^{g(n)} = \infty$ since $g(n)$ is an increasing function that goes to infinity as $n \to \infty$. If $m > -\log_A B$, then

$A^m > \frac{1}{B}$ implying $A^m B > 1$. But m is arbitrarily large, thus we have $m > -\log_A B$. This shows $(A^M B) > 1$, which shows

$$\lim_{n\to\infty} A^{f(n)} B^{g(n)} = \lim_{n\to\infty} |e_z(t_n, t_0)| = \infty.$$

□

2.5 Time Scales with a Graininess Sequence with a Limit of 0

The following theorem deals with isolated time scales where the graininess is defined by a sequence with a limit of 0. We will give a theorem which gives a region of convergence to 0 of the generalized exponential function for a time scale $\mathbb{T} = \{t_0, t_1, \cdots\}$ for which there exists a function $f : [t_0, \infty) \to [0, \infty)$ such that the following hold:

(H1) $f(t_i) = \mu(t_i)$ for all $i \in \mathbb{N}$,

(H2) $\lim_{x\to\infty} f(x) = 0$,

(H3) $f(x)$ is twice differentiable for $x \in (0, \infty)$,

(H4) there exists some $N \in \mathbb{N}$ such that $f'(n) < 0$ for all $n > N$,

(H5) $\lim_{k\to\infty} 2^{k+1} f'(2^k) = 0$,

(H6) $\lim_{k\to\infty} \frac{2^k f''(2^k)}{f'(2^k)} = -2$.

In the following example where $\mu(t_k) = 1/k$, we find a function $f(t)$ such that (H1-H6) all hold. A similar example can be given for $\mu(t_k) = \frac{1}{k\ln(k)}$.

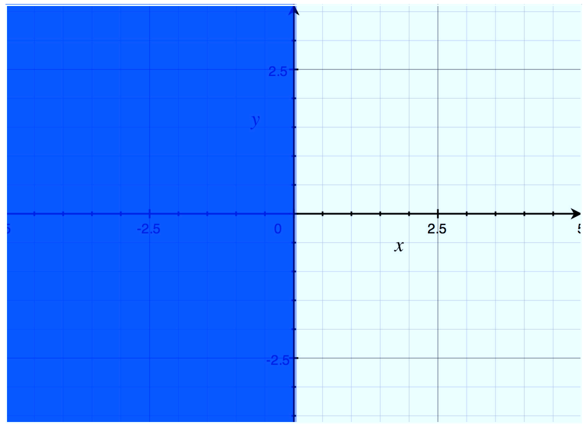

Figure 2.8: Region of convergence of the exponential $e_z(t, t_0)$ to 0 of the generalized exponential function when $\mathbb{T}$ is generated by $\mu(t_k) = \frac{1}{k}$.

Example 2.5.1. Let $t_0 = 0$ and $\mu(t_k) = \frac{1}{k}$ for $k \in \mathbb{N}$. Then choose $f(x) = 1/x$. This satisfies (H1), (H2), and (H3). Note than that $f'(x) = -\frac{1}{x^2}$ is negative for all $x > 0$, so (H4) holds. Furthermore

$$\lim_{k\to\infty} 2^{k+1} f'(2^k) = \lim_{k\to\infty} -\frac{2^{k+1}}{2^{2k}} = \lim_{k\to\infty} -\frac{2}{2^k} = 0,$$

thus (H5) holds. Finally,

$$\lim_{k\to\infty} \frac{2^k f''(2^k)}{f'(2^k)} = \lim_{k\to\infty} -\frac{2^k 2^{2k} 2}{2^{3k}} = -2$$

which satisfies (H6).

The following theorem will show that the generalized exponential function converges to 0 in the left half of the complex plane as seen in Figure 2.8.

Theorem 2.5.1. *Assume* $\mathbb{T} = \{t_0, t_1, \cdots\}$ *where* $t_0 < t_1 < \cdots$ *and* $\sup \mathbb{T} = \infty$ *generated by a graininess defined by* $\mu(t_k)$ *such that* $\lim_{k\to\infty} \mu(t_k) = 0$*. Assume there exists a real-valued function* $f(x)$ *such that the following hypotheses hold:*

(H1) $f(t_i) = \mu(t_i)$ *for all* $i \in \mathbb{N}$*,*

(H2) $\lim_{x\to\infty} f(x) = 0$*,*

(H3) $f(x)$ *is twice differentiable for* $x \in (0, \infty)$*,*

(H4) there exists some $N \in \mathbb{N}$ *such that* $f'(n) < 0$ *for all* $n > N$*,*

(H5) $\lim_{k\to\infty} 2^{k+1} f'(2^k) = 0$*,*

(H6) $\lim_{k\to\infty} \frac{2^k f''(2^k)}{f'(2^k)} = -2$*.*

Let $z \in \mathbb{C} \cap \mathcal{R}$ *be given such that* $\mathrm{Re}(z) < 0$*, then* $\lim_{n\to\infty} e_z(t_n, t_0) = 0$*.*

Proof. Let $z = a + bi$ with $a < 0$ be given. Assume $\lim_{k\to\infty} \mu(t_k) = 0$, and let $f(x)$ be a function such that (H1)-(H6) hold.

By Lemma 2.2.1 we want to show

$$\sum_{k=0}^{\infty} \ln |1 + \mu(t_k)z| = -\sum_{k=0}^{\infty} \ln \left| \frac{1}{1 + \mu(t_k)z} \right|$$

diverges to negative infinity to show convergence of the generalized exponential function to 0. We will do this by using the Cauchy condensation test for divergence, as well as the root test to show divergence.

By the Cauchy condensation test, the sum $\sum_{k=0}^{\infty} \ln \left| \frac{1}{1+f(k)z} \right| = \infty$ if and only if $\sum_{k=0}^{\infty} 2^k \ln \left| \frac{1}{1+f(2^k)z} \right| = \infty$, as long as $\ln \left| \frac{1}{1+f(k)z} \right|$ is eventually non-increasing. To show it is eventually non-increasing, consider the derivative of $\ln\left|\frac{1}{1+f(k)z}\right| = \ln\left|\frac{1}{1+f(k)(a+bi)}\right|$.

$$\begin{aligned}
\frac{d}{dk}\ln\left|\frac{1}{1+f(k)(a+bi)}\right| =& \frac{d}{dk} - \ln|1+f(k)(a+bi)| \\
=& \frac{d}{dk} - \ln\left(\sqrt{(1+af(k))^2+(bf(k))^2)}\right) \\
=& -\frac{1}{2}\frac{2f'(k)\left((a^2+b^2)f(k)+a\right)}{(a^2+b^2)f(k)^2+2af(k)+1} \\
=& -f'(k)\frac{\left((a^2+b^2)f(k)+a\right)}{(a^2+b^2)f(k)^2+2af(k)+1}.
\end{aligned}$$

Let $g(x) = \frac{\left((a^2+b^2)f(x)+a\right)}{(a^2+b^2)f(x)^2+2af(x)+1}$. Note the denominator factors as $\left(f(x) - \frac{-2a+\sqrt{4a^2+4(a^2+b^2)}}{2}\right)\left(f(x) - \frac{-2a-\sqrt{4a^2-4(a^2+b^2)}}{2}\right)$, but $f(x)$ is a real-valued and continuous function on $(0,\infty)$, so there are no real roots of the denominator, thus there are no vertical asymptotes for $f(x)$ with $x \in (0,\infty)$. This implies that $g(x)$ is continuous on the interval $(0,\infty)$. Now consider

$$\begin{aligned}
\lim_{k\to\infty} g(k) = \lim_{k\to\infty} & \frac{\left((a^2+b^2)f(k)+a\right)}{(a^2+b^2)f(k)^2+2af(k)+1} \\
=& \frac{\left((a^2+b^2)\lim_{k\to\infty} f(k)+a\right)}{(a^2+b^2)\lim_{k\to\infty} f(k)^2+2a\lim_{k\to\infty} f(k)+1} \\
=& \frac{0+a}{0+0+1} = a.
\end{aligned}$$

Note $-f'(k)\frac{\left((a^2+b^2)f(k)+a\right)}{(a^2+b^2)f(k)^2+2af(k)+1} = -f'(k)g(k)$. Since $\lim_{k\to\infty} g(k) = a$ and $a < 0$ by assumption, there exists some $K_0 > 0$ such that for all $k > K_0$, $g(k) < 0$. By (H4), there exists some K_1 such that for all $k > K_1$, $f'(k) < 0$. Set $K = \max\{K_0, K_1\}$. We then have $-f'(k)g(k) < 0$ for all $k > K$. Therefore we have the derivative is eventually negative, thus the original term is eventually non-increasing, so we can apply the

Cauchy condensation test. We are now concerned with showing the divergence of

$$\sum_{k=0}^{\infty} 2^k \ln \left| \frac{1}{1+f(2^k)z} \right|.$$

We will now apply the root test to the new summand obtained from the Cauchy condensation test. We will show

$$\lim_{k\to\infty} \sqrt[k]{\left| 2^k \ln \left| \frac{1}{1+f(2^k)z} \right| \right|} = 1$$

and approaches 1 from above, thus showing the summand diverges.

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{\left| 2^k \ln \left| \frac{1}{1+f(2^k)z} \right| \right|} &= \lim_{k\to\infty} \sqrt[k]{\left| -2^k \ln |1+f(2^k)z| \right|} \\
&= \lim_{k\to\infty} \sqrt[k]{\left| 2^k \ln \sqrt{(1+f(2^k)a)^2 + (f(2^k)b)^2} \right|} \\
&= \lim_{k\to\infty} \sqrt[k]{|2^{k-1} \ln ((1+f(2^k)a)^2 + (f(2^k)b)^2)|} \\
&= \lim_{k\to\infty} 2^{1-1/k} \left| \ln \left((1+f(2^k)a)^2 + (f(2^k)b)^2 \right) \right|^{1/k}.
\end{aligned}$$

Clearly the $2^{1-1/k}$ factor will go to 2 as $k \to \infty$, so we will now go on to show the limit exists for the other factors and that they approach $\frac{1}{2}$.

$$\begin{aligned}
&\lim_{k\to\infty} 2^{1-1/k} \left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|^{1/k} \\
&\qquad\qquad = 2\lim_{k\to\infty} \left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|^{1/k} \\
&\qquad\qquad = 2\exp\left(\lim_{k\to\infty} \frac{\ln\left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|}{k}\right).
\end{aligned}$$

The limit can be moved inside the exponent due to the continuity of the exponential. We want to show

$$\lim_{k\to\infty} \frac{\ln\left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|}{k} = -\ln(2),$$

which would imply

$$\lim_{k\to\infty} \sqrt[k]{\left|2^k \ln\left|\frac{1}{1+f(2^k)z}\right|\right|} = 1$$

as desired.

Consider

$$\lim_{k\to\infty} \frac{\ln\left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|}{k}.$$

It can be shown using the continuity of the natural log and (H2), that in the numerator, the argument of the inner natural log approaches 1 from above, so the argument in the outer natural log approaches 0, thus the numerator itself approaches negative infinity as $k \to \infty$. The denominator of the limit we are considering diverges to positive infinity, thus we can apply L'Hopital's rule.

$$\lim_{k\to\infty} \frac{\ln\left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|}{k}$$
$$= \lim_{k\to\infty} \frac{2^{k+1}\ln(2)f'(2^k)}{\ln\left((1+af(2^k))^2+(bf(2^k))^2\right)} \frac{((a^2+b^2)f(2^k)^2+a)}{((a^2+b^2)f(2^k)+2af(2^k)+1)}$$
$$= \lim_{k\to\infty} \frac{2^{k+1}\ln(2)f'(2^k)a}{\ln\left((1+af(2^k))^2+(bf(2^k))^2\right)}. \tag{H2}$$

By (H5), the numerator converges to 0 as $k \to \infty$, and it can be shown using the continuity of the natural log and (H2), that in the denominator the term inside the natural log approaches 1 from above as $k \to \infty$, implying the denominator approaches 0, so we can again apply L'Hopital's rule to obtain

$$\lim_{k\to\infty} \frac{2^{k+1}\ln(2)f'(2^k)a}{\ln\left((1+af(2^k))^2+(bf(2^k))^2\right)} = \lim_{k\to\infty} \frac{2^{k+1}\ln(2)^2 a\left(2^k f''(2^k)+f'(2^k)\right)}{\frac{2^{k+1}\ln(2)f'(2^k)\left((a^2+b^2)f(2^k)+a\right)}{(a^2+b^2)f(2^k)^2+2af(2^k)+1}}.$$

Simplifying, we are left with

$$\lim_{k\to\infty} \ln(2)\left(1+\frac{2^k f''(2^k)}{f'(2^k)}\right) = -\ln(2). \tag{$H6$}$$

Thus

$$\lim_{k\to\infty} \frac{\ln\left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|}{k} = -\ln(2),$$

so

$$2\exp\left(\lim_{k\to\infty} \frac{\ln\left|\ln\left((1+f(2^k)a)^2+(f(2^k)b)^2\right)\right|}{k}\right) = 1,$$

thus

$$\lim_{k\to\infty} \sqrt[k]{2^k \ln\left|\frac{1}{1+f(2^k)z}\right|} = 1.$$

We will now show that

$$\sqrt[k]{2^k \ln \left| \frac{1}{1+f(2^k)z} \right|}$$

eventually approaches 1 from above when $z \in \mathbb{C} \cap \mathcal{R}$ such that $\mathrm{Re}(z) < 0$. We will do this by showing

$$\frac{\ln \left| \ln \left((1+f(2^k)a)^2 + (f(2^k)b)^2 \right) \right|}{k}$$

approaches $-\ln(2)$ from above by looking at its derivative.

$$\frac{d}{dk} \frac{\ln \left| \ln \left((1+f(2^k)a)^2 + (f(2^k)b)^2 \right) \right|}{k}$$
$$= \frac{2^{k+1}\ln(2) f'(2^k) \left((a^2+b^2)f(2^k) + a \right)}{\ln\left((1+af(2^k))^2 + (bf(2^k))^2 \right)} \frac{((a^2+b^2)f(2^k)+a)}{((a^2+b^2)f(2^k) + 2af(2^k) + 1)} - \frac{1}{k}.$$

Note

$$\lim_{k\to\infty} \frac{2^{k+1}\ln(2) f'(2^k) \left((a^2+b^2)f(2^k) + a \right) \left((a^2+b^2)f(2^k) + a \right)}{\ln\left((1+af(2^k))^2 + (bf(2^k))^2 \right) \left((a^2+b^2)f(2^k) + 2af(2^k) + 1 \right)} := h(k) = -\ln(2)$$

from earlier in the proof. Since $-\ln(2) < 0$, we have that there exists a $K > 0$ such that $h(k) < 0$ for all $k > K$. Thus, eventually $h(k) - \frac{1}{k} < 0$ for all $k > K$.

Since $h(k) - \frac{1}{k} < 0$ for all $k > K$, the derivative approaches $-\ln(2)$ from above which implies

$$\lim_{k\to\infty} \sqrt[k]{2^k \ln \left| \frac{1}{1+f(2^k)z} \right|} = 1$$

from above, thus by the root test, we have

$$\sum_{k=0}^{\infty} 2^k \ln \left| \frac{1}{1+f(2^k)z} \right| = \infty.$$

Therefore, via the Cauchy condensation test we have

$$\sum_{k=0}^{\infty} \ln \left| \frac{1}{1 + f(k)z} \right| = \infty,$$

so Lemma 2.2.1 is satisfied showing the generalized exponential function converges to 0 for all $z \in \mathbb{C} \cap \mathcal{R}$ such that $\mathrm{Re}(z) < 0$. $\square$

Chapter 3

Graphs of Regions of Convergence of the Exponential $e_z(t, t_0)$ to 0 in the Complex Plane

The following graphs show the region of convergence of the exponential $e_z(t, t_0)$ to 0 in the complex plane.

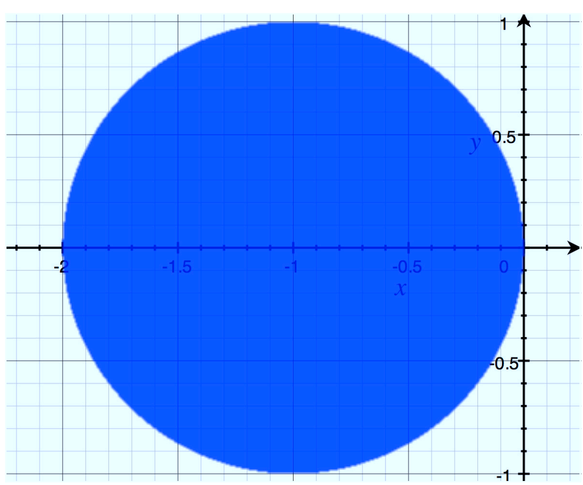

Figure 3.1: Graininess is constantly 1.

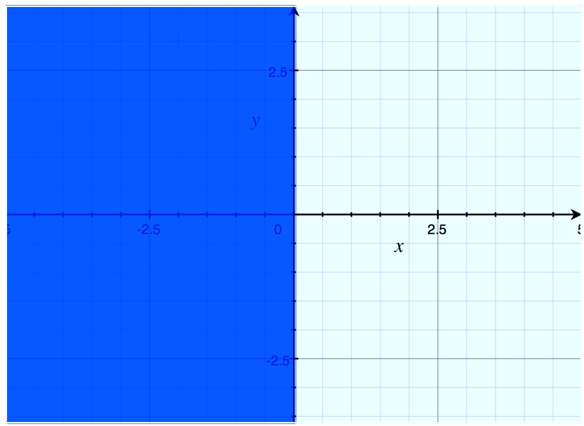

Figure 3.2: Real case, where the graininess is constantly 0.

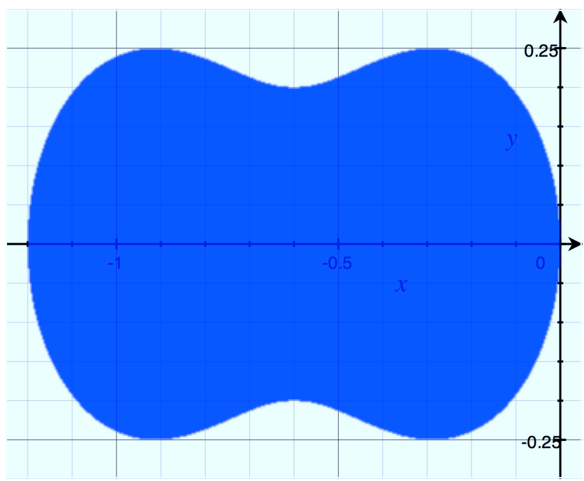

Figure 3.3: Graininess alternating between the values of 1 and 5.

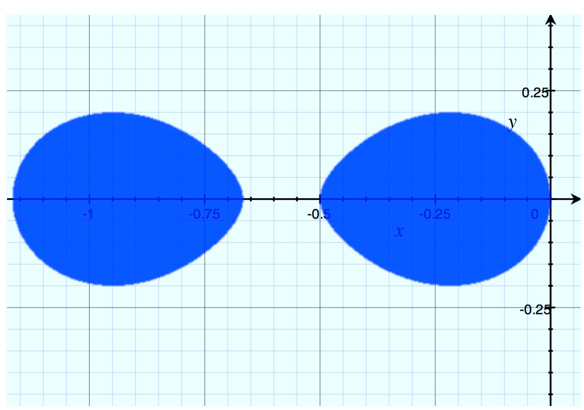

Figure 3.4: Graininess alternating between the values of 1 and 6.

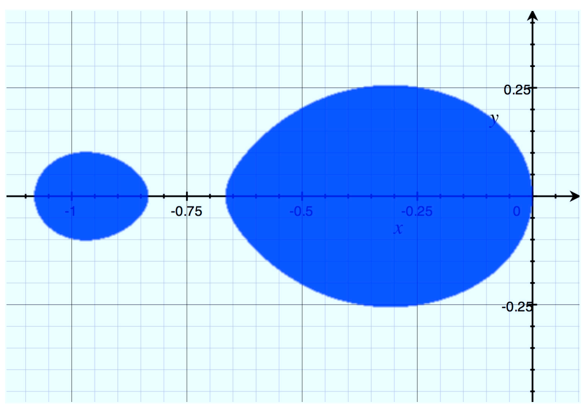

Figure 3.5: Graininess cycling between the values of 1, 3, and 6.

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